附录

正文中式(1) $\sum_{m=1}^{M}\sum_{i=1}^{N_m}\sum_{k=1}^{K}s_{ik}(f(\boldsymbol{x}_i^m)-\mu_k^m)^2=\sum_{m=1}^{M}(\boldsymbol{X}^m\boldsymbol{w}^m)^{\mathrm{T}}\boldsymbol{L}(\boldsymbol{X}^m\boldsymbol{w}^m)$ 的推导过程.

公式中一些字母的表达如下所示：

$$
\begin{cases}
\boldsymbol{F}(\boldsymbol{X}^m)=\boldsymbol{X}^m\boldsymbol{w}^m=[f(\boldsymbol{x}_1^m),\ f(\boldsymbol{x}_2^m),\cdots,f(\boldsymbol{x}_{N_m}^m)]^{\mathrm{T}},\\
\boldsymbol{X}^m=[\boldsymbol{x}_1^m,\boldsymbol{x}_2^m,\cdots,\ \boldsymbol{x}_{N_m}^m]^{\mathrm{T}},\\
\boldsymbol{W}=[\boldsymbol{w}^1,\ \boldsymbol{w}^2,\ \cdots,\ \boldsymbol{w}^M],\\
\boldsymbol{U}^m=[\mu_1^m,\mu_2^m,\cdots,\mu_K^m]^{\mathrm{T}}=\boldsymbol{s}_1\boldsymbol{F}(\boldsymbol{X}^m),\\
\boldsymbol{Q}=[\mathrm{num}_1,\mathrm{num}_2,\cdots,\mathrm{num}_K],\\
\boldsymbol{L}=\boldsymbol{E}-2\boldsymbol{s}\,\boldsymbol{s}_1+\boldsymbol{s}_1^{\mathrm{T}}\boldsymbol{Q}\,\boldsymbol{s}_1,\\
\mathrm{size}(\boldsymbol{s})=N_mK,\mathrm{size}(\boldsymbol{s}_1)=K\,N_m,\\
\boldsymbol{s}_1(k,:)=\dfrac{\boldsymbol{s}(:,k)}{\mathrm{num}_k}.
\end{cases}
$$

证明等式的左边等于等式的右边：

$$
\begin{aligned}
J(\boldsymbol{W})=&\sum_{m=1}^{M}\sum_{i=1}^{N_m}\sum_{k=1}^{K}s_{ik}\,(f(\boldsymbol{x}_i^m)-\mu_k^m)^2=\\
&\sum_{m=1}^{M}\sum_{i=1}^{N_m}\sum_{k=1}^{K}s_{ik}\,[f^2(\boldsymbol{x}_i^m)-2(f(\boldsymbol{x}_i^m)\,\mu_k^m)+(\mu_k^m)^2]^2=\\
&\sum_{m=1}^{M}\sum_{i=1}^{N_m}\sum_{k=1}^{K}s_{ik}\,f^2(\boldsymbol{x}_i^m)-2\sum_{m=1}^{M}\sum_{i=1}^{N_m}\sum_{k=1}^{K}s_{ik}f(\boldsymbol{x}_i^m)\,\mu_k^m+\sum_{m=1}^{M}\sum_{i=1}^{N_m}\sum_{k=1}^{K}s_{ik}\,(\mu_k^m)^2=\\
&J_1(\boldsymbol{W})+J_2(\boldsymbol{W})+J_3(\boldsymbol{W}),
\end{aligned}
$$

分别对 $J_1(\boldsymbol{W})$，$J_2(\boldsymbol{W})$，$J_3(\boldsymbol{W})$ 进行推导：

$$
\begin{aligned}
J_1(\boldsymbol{W})=&\sum_{m=1}^{M}\sum_{i=1}^{N_m}\sum_{k=1}^{K}s_{ik}\,f^2(\boldsymbol{x}_i^m)=\sum_{m=1}^{M}\sum_{i=1}^{N_m}f^2(\boldsymbol{x}_i^m)\sum_{k=1}^{K}s_{ik}=\sum_{m=1}^{M}\sum_{i=1}^{N_m}f^2(\boldsymbol{x}_i^m)=\\
&\sum_{m=1}^{M}\boldsymbol{F}(\boldsymbol{X}^m)^{\mathrm{T}}\boldsymbol{F}(\boldsymbol{X}^m)=\sum_{m=1}^{M}(\boldsymbol{X}^m\boldsymbol{w}^m)^{\mathrm{T}}(\boldsymbol{X}^m\boldsymbol{w}^m),
\end{aligned}
$$

$$
J_2(\boldsymbol{W})=2\sum_{m=1}^{M}\sum_{i=1}^{N_m}\sum_{k=1}^{K}s_{ik}f(\boldsymbol{x}_i^m)\,\mu_k^m=-2\sum_{m=1}^{M}\boldsymbol{F}(\boldsymbol{X}^m)^{\mathrm{T}}\boldsymbol{S}\,\boldsymbol{U}^m,
$$

$$
\begin{aligned}
J_3(\boldsymbol{W})=&\sum_{m=1}^{M}\sum_{i=1}^{N_m}\sum_{k=1}^{K}s_{ik}\,(\mu_k^m)^2=\sum_{m=1}^{M}\sum_{k=1}^{K}(\mu_k^m)^2\sum_{i}^{N_m}s_{ik}=\\
&\sum_{m=1}^{M}\sum_{k=1}^{K}\mathrm{num}_k\,(\mu_k^m)^2=\sum_{m=1}^{M}(\boldsymbol{U}^m)^{\mathrm{T}}\boldsymbol{N}\,\boldsymbol{U}^m,
\end{aligned}
$$

$$
\begin{aligned}
J(\boldsymbol{W})=&J_1(\boldsymbol{W})+J_2(\boldsymbol{W})+J_3(\boldsymbol{W})=\\
&\sum_{m=1}^{M}\boldsymbol{F}(\boldsymbol{X}^m)^{\mathrm{T}}\boldsymbol{F}(\boldsymbol{X}^m)-2\sum_{m=1}^{M}\boldsymbol{F}(\boldsymbol{X}^m)^{\mathrm{T}}\boldsymbol{s}\,\boldsymbol{U}^m+\sum_{m=1}^{M}(\boldsymbol{U}^m)^{\mathrm{T}}\boldsymbol{Q}\,\boldsymbol{U}^m=\\
&\sum_{m=1}^{M}[\boldsymbol{F}(\boldsymbol{X}^m)^{\mathrm{T}}\boldsymbol{F}(\boldsymbol{X}^m)-2\,\boldsymbol{F}(\boldsymbol{X}^m)^{\mathrm{T}}\boldsymbol{s}\,\boldsymbol{U}^m+(\boldsymbol{U}^m)^{\mathrm{T}}\boldsymbol{Q}\,\boldsymbol{U}^m]=\\
&\sum_{m=1}^{M}[(\boldsymbol{X}^m\boldsymbol{w}^m)^{\mathrm{T}}(\boldsymbol{E}-2\boldsymbol{s}\,\boldsymbol{s}_1+\boldsymbol{s}_1^{\mathrm{T}}\boldsymbol{Q}\,\boldsymbol{s}_1)(\boldsymbol{X}^m\boldsymbol{w}^m)]=\\
&\sum_{m=1}^{M}[(\boldsymbol{X}^m\boldsymbol{w}^m)^{\mathrm{T}}\boldsymbol{L}(\boldsymbol{X}^m\boldsymbol{w}^m)].
\end{aligned}
$$

得证.