基于一致系数求积的广义牛顿法求解非线性方程组

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摘要: 在本文中,我们采用一致系数求积公式近似逼近泰勒余项,得到一种新的求解非线性方程组的广义牛顿法. 首先,我们给出算法的一般形式,然后证明算法是三阶收敛的,并且在一定的温和条件下可以达到五阶收敛.最后,给出数值例子说明算法的有效性和稳定性.

关键词**:** 一致系数求积; 广义牛顿法; 非线性方程组; 三阶收敛;

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考虑如下非线性方程组：

$$\mathbf{f}(\mathbf{x}) := (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))^T = \mathbf{0}, \tag{1}$$

其中$\mathbf{x} = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, $\mathbf{f} : \mathcal{K} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ 在凸集$\mathcal{K}$ 中是充分Fréchet可微的.众所周知,牛顿法是求解非线性方程组(1) 的经典迭代法.而近几年,出现了许多利用Adomian分解,同伦摄动,牛顿柯特斯求积等技巧求解非线性方程组(1) 的牛顿衍生算法,这些方法相较于牛顿法具有更高的收敛性,可参考文献[1], [2], [3], [4], [5], [6], [7], [9]等. 在本文中,我们提出一种新的广义牛顿迭代算法,即采用基于一致系数求积的牛顿法求解非线性方程组(1). 此广义牛顿迭代法只需要估计$\mathbf{f}(\mathbf{x})$ 以及它在不同点处的雅可比矩阵即可.数值实验表明, 这个迭代算法是三次收敛的并且在一定的温和条件下可达到五阶收敛.

1　预备知识

一致系数求积公式[10]　定义如下形式的积分

$$I(f) = \int_a^b \rho(x) f(x) dx, \tag{2}$$

这里$\rho(x)$是权函数,通常(2)式的不定积分无法求解,为此我们需要定义能够近似$I(f)$的数值积分公式.典型的数值积分公式具有如下形式

$$I_n(f) = \sum_{k=1}^{n} A_k f(x_k),$$

其中x_k称为求积公式的节点,A_k 称为求积系数.假设$\rho(x) = 1$,则n个节点的一致系数求积公式具有如下简单形式

$$I_n(f) = A_n \sum_{k=1}^{n} f(x_k), \tag{3}$$

其中$A_n = \frac{b-a}{n}$,节点x_k满足

$$A_n \sum_{k=1}^{n} x_k^j = \frac{1}{j+1}(b^{j+1} - a^{j+1}), \quad j = 1, 2, ..., n. \tag{4}$$

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由[10]知,节点x_k是关于积分区间的中点$\frac{a+b}{2}$对称的.

对任意的$\mathbf{x},\mathbf{x}^s\in\mathcal{K}$, 由泰勒定理 [8] 得

$$\mathbf{f}(\mathbf{x})=\mathbf{f}(\mathbf{x}^s)+\int_0^1\mathbf{f}'(\mathbf{x}^s+w(\mathbf{x}-\mathbf{x}^s))(\mathbf{x}-\mathbf{x}^s)dw, \tag{5}$$

其中$\mathbf{f}'(\mathbf{x})$ 为函数$\mathbf{f}$ 在点$\mathbf{x}\in\mathbb{R}^n$ 处的雅克比矩阵. 通过积分变换我们有

$$\int_0^1\mathbf{f}'(\mathbf{x}^s+w(\mathbf{x}-\mathbf{x}^s))(\mathbf{x}-\mathbf{x}^s)dw=\frac{1}{2}\int_{-1}^1\mathbf{f}'\Big(\frac{1}{2}w(\mathbf{x}-\mathbf{x}^s)+\frac{1}{2}(\mathbf{x}+\mathbf{x}^s)\Big)(\mathbf{x}-\mathbf{x}^s)dw.$$

应用M个节点的一致系数求积公式,我们得到如下的数值积分逼近:

$$\frac{1}{2}\int_{-1}^1\mathbf{f}'\Big(\frac{1}{2}w(\mathbf{x}-\mathbf{x}^s)+\frac{1}{2}(\mathbf{x}+\mathbf{x}^s)\Big)(\mathbf{x}-\mathbf{x}^s)dw\approx\frac{1}{2}\Big[\sum_{k=1}^M\tau_M\mathbf{f}'\Big(\mathbf{v}_{M,k}(\mathbf{x})\Big)\Big](\mathbf{x}-\mathbf{x}^s). \tag{6}$$

其中

$$\mathbf{v}_{M,k}(\mathbf{x}):=\frac{1}{2}w_{M,k}(\mathbf{x}-\mathbf{x}^s)+\frac{1}{2}(\mathbf{x}+\mathbf{x}^s),\quad k=1,2,...,M\tau_M=\frac{2}{M}$$

节点$w_{M,k}$表示M个节点中的第k个节点,则由节点关于原点对称，满足$\sum_{k=1}^M w_{M,k}=0$.

备注**1**　我们知道M 个节点的一致系数公式具有至少M次代数精度,并且M为偶数时,代数精度为$M+1$. 因此,当$M\geq 2$时,有

$$\frac{1}{3}=\int_{-1}^1\frac{1}{2}w^2dw=\sum_{t=1}^M\tau_M\frac{w_{M,t}^2}{2}, \tag{7}$$

以及

$$0=\int_{-1}^1\frac{1}{2}w^3dw=\sum_{t=1}^M\tau_M\frac{w_{M,t}^3}{2}. \tag{8}$$

成立.

备注**2**　如果能建立求积公式(3),那么必须要求(4)中的节点$x_k,k=1,2,...,n$为不同的实数,而当$n=8$以及$n\geq 10$时, 解得(4)中总会出现一对复共轭节点.因此,在这些情形下无法构造一致系数公式.

2　算法及其收敛性分析

基于以上讨论,我们给出一种新的基于一致系数求积公式求解非线性方程组(1)的广义牛顿迭代算法.

算法**1.** (基于一致系数求积公式的广义牛顿法)

Step 0. 给定初值$\mathbf{x}^0\in\mathbb{R}^n$ 以及正整数M. $s:=0$.

Step 1. 计算

$$\mathbf{y}^s=\mathbf{x}^s-\mathbf{f}'(\mathbf{x}^s)^{-1}\mathbf{f}(\mathbf{x}^s).$$

Step 2. 计算

$$\mathbf{x}^{s+1}=\mathbf{x}^s-\Big[\sum_{k=1}^M\theta_{M,k}\mathbf{f}'\Big(\frac{1}{2}w_{M,k}(\mathbf{x}-\mathbf{y}^s)+\frac{1}{2}(\mathbf{x}+\mathbf{y}^s)\Big)\Big]^{-1}\mathbf{f}(\mathbf{x}^s),$$

其中$\theta_{M,k}\geq 0,\quad \theta_{M,k}=\theta_{M,(M-k+1)},\quad \sum\limits_{k=1}^M\theta_{M,k}=1.$

Step 3. $s:=s+1$ 返回到**Step 1**.

由于节点$\{w_{M,k}\}$ 是关于原点对称的，并且当M为奇数时，原点必是节点，加上系数$\theta_{M,k}$的取法，可得如下关系式

$$\sum_{k=1}^{M}\theta_{M,k}w_{M,k}=0. \tag{9}$$

下面我们说明,上述建立的求解非线性方程组(1)的算法1是局部三次收敛的. 假设$\boldsymbol{\alpha}\in\mathbb{R}^n$ 是非线性方程组(1)的一个解, $\mathbf{f}$ 在$\boldsymbol{\alpha}$的一个邻域$\mathcal{N}(\boldsymbol{\alpha})$ 内是充分连续可微的.对任意的$\mathbf{x}\in\mathcal{N}(\boldsymbol{\alpha})$, $J(\mathbf{x})\in\mathbb{R}^{n\times n}$为$\mathbf{f}$在点$\mathbf{x}$ 处的雅克比矩阵,即

$$J_{ij}(\mathbf{x})=\frac{\partial f_i(\mathbf{x})}{\partial x_j}. \tag{10}$$

这里我们还假设$J(\mathbf{x})$ 对任意$\mathbf{x}\in\mathcal{N}(\boldsymbol{\alpha})$ 都是非奇异的,并且定义$H(\mathbf{x})$ 为$J(\mathbf{x})$ 的逆矩阵.那么我们得到

$$\sum_{j=1}^{n}H_{ij}(\mathbf{x})J_{jk}(\mathbf{x})=\delta_{ik}, \tag{11}$$

其中δ_{ik} 是克罗内克符号函数. 对等式(11)两边相对于x_l 求微分,我们可以得到

$$\sum_{j=1}^{n}\frac{\partial H_{ij}(\mathbf{x})}{\partial x_l}\frac{\partial f_j(\mathbf{x})}{\partial x_k}=-\sum_{j=1}^{n}H_{ij}(\mathbf{x})\frac{\partial^2 f_j(\mathbf{x})}{\partial x_k\partial x_l},\quad i,k,l\in\{1,\dots,n\}. \tag{12}$$

下面我们给出所需的两个引理,类似于文献[3]的证明过程,引理可以很容易的得到.

引理**1.** 对任意$\mathbf{x}\in\mathcal{N}(\boldsymbol{\alpha})$, 定义$\mathbf{y}(\mathbf{x})=(y_1(\mathbf{x}),\dots,y_n(\mathbf{x}))\in\mathbb{R}^n$ 为经典牛顿法的迭代函数:

$$y_i(\mathbf{x})=x_i-\sum_{j=1}^{n}H_{ij}(\mathbf{x})f_j(\mathbf{x}),\qquad i=1,\dots,n.$$

那么我们有

$$\frac{\partial y_i(\mathbf{x})}{\partial x_k}=-\sum_{j=1}^{n}\frac{\partial H_{ij}(\mathbf{x})}{\partial x_k}f_j(\mathbf{x}),$$

和

$$\frac{\partial^2 y_i(\mathbf{x})}{\partial x_k\partial x_l}=-\sum_{j=1}^{n}\frac{\partial^2 H_{ij}(\mathbf{x})}{\partial x_k\partial x_l}f_j(\mathbf{x})-\sum_{j=1}^{n}\frac{\partial H_{ij}(\mathbf{x})}{\partial x_k}\frac{\partial f_j(\mathbf{x})}{\partial x_l},$$

$i,k,l\in\{1,2,\dots,n\}$成立. 特殊的有

$$\frac{\partial y_i(\boldsymbol{\alpha})}{\partial x_k}=0\quad 和\quad \frac{\partial^2 y_i(\boldsymbol{\alpha})}{\partial x_k\partial x_l}=\sum_{j=1}^{n}H_{ij}(\boldsymbol{\alpha})\frac{\partial^2 f_j(\boldsymbol{\alpha})}{\partial x_l\partial x_k}.$$

引理**2.** 对任意$\mathbf{x}\in\mathcal{N}(\boldsymbol{\alpha})$, $\mathbf{u}_i^M(\mathbf{x})=\frac{1}{2}w_{M,i}(\mathbf{y}(\mathbf{x})-\mathbf{x})+\frac{1}{2}(\mathbf{y}(\mathbf{x})+\mathbf{x})$ 为迭代函数,使得

$$u_{ij}^M(\mathbf{x})=\frac{1}{2}w_{M,i}(y_j(\mathbf{x})-x_j)+\frac{1}{2}(y_j(\mathbf{x})+x_j), \tag{13}$$

$j=1,2,\dots,n$, $i=1,2,\dots,M$, 其中$\mathbf{u}_i^M(\mathbf{x})=(u_{i1}^M(\mathbf{x}),u_{i2}^M(\mathbf{x}),...,u_{in}^M(\mathbf{x}))^T\in\mathbb{R}^n$. 那么有等式

$$\frac{\partial u_{ij}^M(\boldsymbol{\alpha})}{\partial x_k}=\frac{1-w_{M,i}}{2}\delta_{jk},$$

和

$$\frac{\partial^2 u_{ij}^M(\boldsymbol{\alpha})}{\partial x_k\partial x_l}=\frac{1+w_{M,i}}{2}\sum_{t=1}^{M}H_{jt}(\boldsymbol{\alpha})\frac{\partial^2 f_t(\boldsymbol{\alpha})}{\partial x_l\partial x_k},$$

$i\in\{1,2,\dots,M\}$,$j,k,l\in\{1,2,\dots,n\}$成立.

接下来,我们给出算法1 的局部三次收敛性定理.

定理1 函数$\mathbf{f}:\mathcal{N}(\boldsymbol{\alpha})\subseteq\mathbb{R}^n\to\mathbb{R}^n$ 在$\boldsymbol{\alpha}\in\mathbb{R}^n$的一个邻域$\mathcal{N}(\boldsymbol{\alpha})$内是充分可微的, $\boldsymbol{\alpha}$ 是非线性方程组(1)的一个解. 选取充分接近解$\boldsymbol{\alpha}$的初始值$\mathbf{x}^0$ 并且假设$J(\mathbf{x})$ 在邻域$\mathcal{N}(\boldsymbol{\alpha})$内是连续非奇异的. 那么算法1产生的序列$\{\mathbf{x}^k\}$是三次收敛到$\boldsymbol{\alpha}$ 的.

证明: 任意给定$\mathbf{x}\in\mathcal{N}(\boldsymbol{\alpha})$, 定义$W(\mathbf{x})=Z^{-1}(\mathbf{x})$, 这里

$$Z(\mathbf{x})=\sum_{t=1}^{M}\theta_{M,t}J\Big(\frac{1}{2}w_{M,t}(\mathbf{y}(\mathbf{x})-\mathbf{x})+\frac{1}{2}(\mathbf{y}(\mathbf{x})+\mathbf{x})\Big)=\sum_{t=1}^{M}\theta_{M,t}J(\mathbf{u}_t^M(\mathbf{x})). \tag{14}$$

对每一个$\mathbf{x}\in\mathbb{R}^n$, 定义$\mathbf{g}(\mathbf{x}):=(g_1(\mathbf{x}),\ldots,g_n(\mathbf{x}))^T$,其中

$$g_j(\mathbf{x})=y_j(\mathbf{x})+\sum_{q=1}^{n}H_{jq}(\mathbf{x})f_q(\mathbf{x})-\sum_{q=1}^{n}W_{jq}(\mathbf{x})f_q(\mathbf{x}). \tag{15}$$

对每一个$j\in\{1,\ldots,n\}$, $g_j(\mathbf{x})$ 在点$\boldsymbol{\alpha}$ 处泰勒展开

$$g_j(\mathbf{x})=g_j(\boldsymbol{\alpha})+\sum_{j_1=1}^{n}\frac{\partial g_j(\boldsymbol{\alpha})}{\partial x_{j_1}}e_{j_1}+\frac{1}{2}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\frac{\partial^2 g_j(\boldsymbol{\alpha})}{\partial x_{j_1}\partial x_{j_2}}e_{j_1}e_{j_2}+\cdots, \tag{16}$$

其中$e_{j_k}=x_{j_k}-\alpha_{j_k}$.

首先,我们证明$\partial g_j(\boldsymbol{\alpha})/\partial x_k=0,\ j,k\in\{1,2,\ldots,n\}$. (15)式两边同时左乘$Z_{ij}(\mathbf{x})$,

$$Z_{ij}(\mathbf{x})\Big(g_j(\mathbf{x})-y_j(\mathbf{x})-\sum_{q=1}^{n}H_{jq}(\mathbf{x})f_q(\mathbf{x})\Big)+Z_{ij}(\mathbf{x})\Big(\sum_{q=1}^{n}W_{jq}(\mathbf{x})f_q(\mathbf{x})\Big)=0,$$

$i,j\in\{1,2,\ldots,n\}$. 那么

$$\sum_{j=1}^{n}Z_{ij}\Big(g_j(\mathbf{x})-y_j(\mathbf{x})-\sum_{q=1}^{n}H_{jq}(\mathbf{x})f_q(\mathbf{x})\Big)+\sum_{j=1}^{n}Z_{ij}(\mathbf{x})\Big(\sum_{q=1}^{n}W_{jq}(\mathbf{x})f_q(\mathbf{x})\Big)=0,$$

$i\in\{1,2,\ldots,n\}$. 应用$W(\mathbf{x})=Z^{-1}(\mathbf{x})$,等式简化为

$$\sum_{j=1}^{n}Z_{ij}(\mathbf{x})\Big(g_j(\mathbf{x})-y_j(\mathbf{x})-\sum_{q=1}^{n}H_{jq}(\mathbf{x})f_q(\mathbf{x})\Big)+f_i(\mathbf{x})=0, i\in\{1,2,\ldots,n\}. \tag{17}$$

对(17)式两边相对于x_k求微分,应用(11) 和引理1可得

$$\begin{aligned}0 &= \frac{\partial f_i(\mathbf{x})}{\partial x_k}+\sum_{j=1}^{n}\frac{\partial Z_{ij}(\mathbf{x})}{\partial x_k}\Big(g_j(\mathbf{x})-y_j(\mathbf{x})-\sum_{q=1}^{n}H_{jq}(\mathbf{x})f_q(\mathbf{x})\Big)\\ &\quad+\sum_{j=1}^{n}Z_{ij}(\mathbf{x})\Big(\frac{\partial g_j(\mathbf{x})}{\partial x_k}-\frac{\partial y_j(\mathbf{x})}{\partial x_k}-\sum_{q=1}^{n}\frac{\partial H_{jq}(\mathbf{x})}{\partial x_k}f_q(\mathbf{x})-\sum_{q=1}^{n}H_{jq}(\mathbf{x})\frac{\partial f_q(\mathbf{x})}{\partial x_k}\Big)\\ &= \frac{\partial f_i(\mathbf{x})}{\partial x_k}+\sum_{j=1}^{n}\frac{\partial Z_{ij}(\mathbf{x})}{\partial x_k}\Big(g_j(\mathbf{x})-y_j(\mathbf{x})-\sum_{q=1}^{n}H_{jq}(\mathbf{x})f_q(\mathbf{x})\Big)\\ &\quad+\sum_{j=1}^{n}Z_{ij}(\mathbf{x})\Big(\frac{\partial g_j(\mathbf{x})}{\partial x_k}-\delta_{jk}\Big),\end{aligned} \tag{18}$$

上式中令$\mathbf{x}=\boldsymbol{\alpha}$,应用(10)得

$$\sum_{j=1}^{n}J_{ij}(\boldsymbol{\alpha})\frac{\partial g_j(\boldsymbol{\alpha})}{\partial x_k}=J_{ik}(\boldsymbol{\alpha})-\frac{\partial f_i(\boldsymbol{\alpha})}{\partial x_k}=0.$$

由假设$J(\mathbf{x})$ 对所有$\mathbf{x}\in\mathcal{N}(\boldsymbol{\alpha})$都是非奇异的, 所以有

$$\frac{\partial g_j(\boldsymbol{\alpha})}{\partial x_k}=0,\qquad j,k\in\{1,2,\ldots,n\}. \tag{19}$$

接下来,我们证明$\partial^2 g_j(\boldsymbol{\alpha})/(\partial x_k \partial x_l) = 0,\ j,k,l \in \{1,2,\dots,n\}$. 对(18)式两边相对于$x_l$取微分,应用(11) 和引理1 ,我们得到

$$\begin{aligned}
0 &= \frac{\partial^2 f_i(\mathbf{x})}{\partial x_k \partial x_l} + \sum_{j=1}^{n} \frac{\partial^2 Z_{ij}(\mathbf{x})}{\partial x_k \partial x_l}\Big(g_j(\mathbf{x}) - y_j(\mathbf{x}) - \sum_{q=1}^{n} H_{jq}(\mathbf{x}) f_q(\mathbf{x})\Big) \\
&\quad + \sum_{j=1}^{n} \frac{\partial Z_{ij}(\mathbf{x})}{\partial x_k}\Big(\frac{\partial g_j(\mathbf{x})}{\partial x_l} - \frac{\partial y_j(\mathbf{x})}{\partial x_l} - \sum_{q=1}^{n} \frac{\partial H_{jq}(\mathbf{x})}{\partial x_l} f_q(\mathbf{x}) - \sum_{q=1}^{n} H_{jq}(\mathbf{x}) \frac{\partial f_q(\mathbf{x})}{\partial x_l}\Big) \\
&\quad + \sum_{j=1}^{n} \frac{\partial Z_{ij}(\mathbf{x})}{\partial x_l}\Big(\frac{\partial g_j(\mathbf{x})}{\partial x_k} - \delta_{jk}\Big) + \sum_{j=1}^{n} Z_{ij}(\mathbf{x}) \frac{\partial^2 g_j(\mathbf{x})}{\partial x_k \partial x_l} \\
&= \frac{\partial^2 f_i(\mathbf{x})}{\partial x_k \partial x_l} + \sum_{j=1}^{n} \frac{\partial^2 Z_{ij}(\mathbf{x})}{\partial x_k \partial x_l}\Big(g_j(\mathbf{x}) - y_j(\mathbf{x}) - \sum_{q=1}^{n} H_{jq}(\mathbf{x}) f_q(\mathbf{x})\Big) + \sum_{j=1}^{n} \frac{\partial Z_{ij}(\mathbf{x})}{\partial x_k}\Big(\frac{\partial g_j(\mathbf{x})}{\partial x_l} - \delta_{jl}\Big) \\
&\quad + \sum_{j=1}^{n} \frac{\partial Z_{ij}(\mathbf{x})}{\partial x_l}\Big(\frac{\partial g_j(\mathbf{x})}{\partial x_k} - \delta_{jk}\Big) + \sum_{j=1}^{n} Z_{ij}(\mathbf{x}) \frac{\partial^2 g_j(\mathbf{x})}{\partial x_k \partial x_l}, \qquad (20)
\end{aligned}$$

令$\mathbf{x} = \boldsymbol{\alpha}$并将(19)代入到(20)得,

$$-\frac{\partial Z_{il}(\boldsymbol{\alpha})}{\partial x_k} - \frac{\partial Z_{ik}(\boldsymbol{\alpha})}{\partial x_l} + \sum_{j=1}^{n} J_{ij}(\boldsymbol{\alpha}) \frac{\partial^2 g_j(\boldsymbol{\alpha})}{\partial x_k \partial x_l} + \frac{\partial^2 f_i(\boldsymbol{\alpha})}{\partial x_k \partial x_l} = 0. \qquad (21)$$

另一方面,根据(14), 我们有

$$Z_{ij}(\mathbf{x}) = \sum_{t=1}^{M} \theta_{M,t} J_{ij}(\mathbf{u}_t^M(\mathbf{x})) = \sum_{t=1}^{M} \theta_{M,t} \frac{\partial f_i}{\partial u_{tj}^M}(\mathbf{u}_t^M(\mathbf{x})). \qquad (22)$$

对方程(22)两边相对于x_k取微分得到,

$$\frac{\partial Z_{ij}(\mathbf{x})}{\partial x_k} = \sum_{t=1}^{M} \theta_{M,t}\Big(\sum_{p=1}^{M} \frac{\partial^2 f_i(\mathbf{u}_t^M(\mathbf{x}))}{\partial u_{tj}^M \partial u_{tp}^M} \frac{\partial u_{tp}^M(\mathbf{x})}{\partial x_k}\Big). \qquad (23)$$

令$\mathbf{x} = \boldsymbol{\alpha}$ 并应用引理2,(23)式为

$$\begin{aligned}
\frac{\partial Z_{ij}(\boldsymbol{\alpha})}{\partial x_k} &= \sum_{t=1}^{M} \theta_{M,t}\Big(\sum_{p=1}^{n} \frac{\partial^2 f_i(\boldsymbol{\alpha})}{\partial x_j \partial x_p} \frac{1 - w_{M,t}}{2} \delta_{pk}\Big) \\
&= \sum_{t=1}^{M} \frac{\theta_{M,t}(1 - w_{M,t})}{2} \frac{\partial^2 f_i(\boldsymbol{\alpha})}{\partial x_j \partial x_k}. \qquad (24)
\end{aligned}$$

由于函数$f_i(\mathbf{x})$是充分可微的, 将(24)式代入到(21)式满足

$$-\sum_{t=1}^{M} \theta_{M,t}(1 - w_{M,t}) \frac{\partial^2 f_i(\boldsymbol{\alpha})}{\partial x_k \partial x_l} + \sum_{j=1}^{n} J_{ij}(\boldsymbol{\alpha}) \frac{\partial^2 g_j(\boldsymbol{\alpha})}{\partial x_k \partial x_l} + \frac{\partial^2 f_i(\boldsymbol{\alpha})}{\partial x_k \partial x_l} = 0.$$

结合式子(9),有

$$\sum_{j=1}^{n} J_{ij}(\boldsymbol{\alpha}) \frac{\partial^2 g_j(\boldsymbol{\alpha})}{\partial x_k \partial x_l} = 0.$$

由假设$J(\boldsymbol{\alpha})$是非奇异的, 所以我们有

$$\frac{\partial^2 g_j(\boldsymbol{\alpha})}{\partial x_k \partial x_l} = 0, \qquad j,k,l \in \{1,2,\dots,n\}. \qquad (25)$$

因此, 由(16), (19), 和(25) 可知算法1 是三次收敛的.

定理**2**　如果定理1中的假设成立, $M \geq 2$并且$\mathbf{f}$的坐标函数$\{f_i\}$ 满足$\partial^2 f_i(\boldsymbol{\alpha})/(\partial x_j \partial x_k) = 0$, $i,j,k \in \{1,2,\ldots,n\}$,那么算法1产生的序列$\{\mathbf{x}^k\}$是五阶收敛到$\boldsymbol{\alpha}$的.

证明:　我们首先证明$\partial^3 g_j(\boldsymbol{\alpha})/(\partial x_k \partial x_l \partial x_m) = 0$, $j,k,l,m \in \{1,2,\ldots,n\}$. 对(23)式两边相对于$x_l$取微分,并令$\mathbf{x} = \boldsymbol{\alpha}$,应用引理2计算可得

$$\begin{aligned}\frac{\partial^2 Z_{ij}(\boldsymbol{\alpha})}{\partial x_k \partial x_l} &= \sum_{t=1}^{M} \frac{\theta_{M,t}(1-w_{M,t})^2}{4} \frac{\partial^3 f_i(\boldsymbol{\alpha})}{\partial x_j \partial x_k \partial x_l} \\ &\quad + \sum_{t=1}^{M} \frac{\theta_{M,t}(1+w_{M,t})}{2} \sum_{p=1}^{n} \frac{\partial^2 f_i(\boldsymbol{\alpha})}{\partial x_j \partial x_p} \Big(\sum_{q=1}^{n} H_{pq}(\boldsymbol{\alpha}) \frac{\partial^2 f_q(\boldsymbol{\alpha})}{\partial x_l \partial x_k} \Big). \end{aligned} \tag{26}$$

另一方面,对(20)式两边相对于x_m 取微分,然后令$\mathbf{x} = \boldsymbol{\alpha}$由(11),(19) 和(25)和引理1我们得到

$$-\frac{\partial^2 Z_{im}(\boldsymbol{\alpha})}{\partial x_k \partial x_l} - \frac{\partial^2 Z_{il}(\boldsymbol{\alpha})}{\partial x_k \partial x_m} - \frac{\partial^2 Z_{ik}(\boldsymbol{\alpha})}{\partial x_l \partial x_m} + \sum_{j=1}^{n} J_{ij}(\boldsymbol{\alpha}) \frac{\partial^3 g_j(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_m} + \frac{\partial^3 f_i(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_m} = 0. \tag{27}$$

将(26)式代入到(27)式有

$$\begin{aligned} 0 &= \Big(1 - \frac{3}{4} \sum_{t=1}^{N} \theta_{M,t}(1-w_{M,t})^2\Big) \frac{\partial^3 f_i(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_m} + \sum_{j=1}^{n} J_{ij}(\boldsymbol{\alpha}) \frac{\partial^3 g_j(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_m} \\ &\quad -\frac{1}{2} \sum_{t=1}^{M} \theta_{M,t}(1+w_{M,t}) \Big(\sum_{p=1}^{n} \frac{\partial^2 f_i(\boldsymbol{\alpha})}{\partial x_m \partial x_p} \sum_{q=1}^{n} H_{pq}(\boldsymbol{\alpha}) \frac{\partial^2 f_q(\boldsymbol{\alpha})}{\partial x_l \partial x_k} \Big) \\ &\quad -\frac{1}{2} \sum_{t=1}^{M} \theta_{M,t}(1+w_{M,t}) \Big(\sum_{p=1}^{n} \frac{\partial^2 f_i(\boldsymbol{\alpha})}{\partial x_l \partial x_p} \sum_{q=1}^{n} H_{pq}(\boldsymbol{\alpha}) \frac{\partial^2 f_q(\boldsymbol{\alpha})}{\partial x_m \partial x_k} \Big) \\ &\quad -\frac{1}{2} \sum_{t=1}^{M} \theta_{M,t}(1+w_{M,t}) \Big(\sum_{p=1}^{n} \frac{\partial^2 f_i(\boldsymbol{\alpha})}{\partial x_k \partial x_p} \sum_{q=1}^{n} H_{pq}(\boldsymbol{\alpha}) \frac{\partial^2 f_q(\boldsymbol{\alpha})}{\partial x_m \partial x_l} \Big). \end{aligned} \tag{28}$$

由假设$M \geq 2$,$\partial^2 f_i(\boldsymbol{\alpha})/(\partial x_j \partial x_k) = 0$,$i,j,k \in \{1,2,\ldots,n\}$, 通过(7), (9)和(28)我们有

$$\sum_{j=1}^{n} J_{ij}(\boldsymbol{\alpha}) \frac{\partial^3 g_j(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_m} = 0.$$

由$J(\boldsymbol{\alpha})$的非奇异性, 可得

$$\frac{\partial^3 g_j(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_m} = 0, \quad j,k,l,m \in \{1,2,\ldots,n\}. \tag{29}$$

类似的,我们证明$\partial^4 g_j(\boldsymbol{\alpha})/(\partial x_k \partial x_l \partial x_m \partial x_r) = 0$, $j,k,l,m,r \in \{1,2,\ldots,n\}$.对(23)式两边求二次微分,然后令$\mathbf{x} = \boldsymbol{\alpha}$应用(7), (8),引理2和假设$\partial^2 f_i(\boldsymbol{\alpha})/(\partial x_j \partial x_k) = 0$, $i,j,k \in \{1,2,\ldots,n\}$, 我们有

$$\begin{aligned} \frac{\partial^3 Z_{ij}(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_m} &= \sum_{t=1}^{M} \frac{\theta_{M,t}(1-w_{M,t})^3}{8} \frac{\partial^4 f_i(\boldsymbol{\alpha})}{\partial x_j \partial x_k \partial x_l \partial x_m} \\ &= \sum_{t=1}^{M} \frac{\theta_{M,t}}{8} \Big(1 - 3w_{M,t} + 3w_{M,t}^2 - w_{M,t}^3\Big) \frac{\partial^4 f_i(\boldsymbol{\alpha})}{\partial x_j \partial x_k \partial x_l \partial x_m} \\ &= \frac{1}{4} \cdot \frac{\partial^4 f_i(\boldsymbol{\alpha})}{\partial x_j \partial x_k \partial x_l \partial x_m}. \end{aligned} \tag{30}$$

另一方面,对(20)式两边求二次微分,并令$\mathbf{x} = \boldsymbol{\alpha}$ 应用(19), (25), (29) 和引理1有

$$\begin{aligned} 0 &= \frac{\partial^4 f_i(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_m \partial x_r} + \sum_{j=1}^{n} J_{ij}(\boldsymbol{\alpha}) \frac{\partial^4 g_j(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_m \partial x_r} \\ &\quad - \frac{\partial^3 Z_{ir}(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_m} - \frac{\partial^3 Z_{im}(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_r} - \frac{\partial^3 Z_{il}(\boldsymbol{\alpha})}{\partial x_k \partial x_m \partial x_r} - \frac{\partial^3 Z_{ik}(\boldsymbol{\alpha})}{\partial x_l \partial x_m \partial x_r}. \end{aligned} \tag{31}$$

我们将(30)式代入到(31)式中,可得

$$\sum_{j=1}^{n} J_{ij}(\boldsymbol{\alpha})\frac{\partial^4 g_j(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_m \partial x_r} = 0,$$

由$J(\boldsymbol{\alpha})$的非奇异性, 有

$$\frac{\partial^4 g_j(\boldsymbol{\alpha})}{\partial x_k \partial x_l \partial x_m \partial x_r} = 0, \quad j,k,l,m,r \in \{1,2,\ldots,n\}. \tag{32}$$

由(16), (19), (25), (29)和(32)可得算法1 是五阶收敛的.

3 数值实验

下面例子1、2和3中,实验停止的标准设为

$$\|\mathbf{f}(\mathbf{x}^s)\|_\infty + \|\mathbf{x}^{s+1} - \mathbf{x}^s\|_\infty \le \epsilon,$$

其中$\|\cdot\|_\infty$ 指无穷范数. 如文献[3]中一样, 方法的收敛阶p 由如下的近似值给出

$$p \approx \frac{\ln(\|\mathbf{x}^{s+1} - \mathbf{x}^s\|_\infty / \|\mathbf{x}^s - \mathbf{x}^{s-1}\|_\infty)}{\ln(\|\mathbf{x}^s - \mathbf{x}^{s-1}\|_\infty / \|\mathbf{x}^{s-1} - \mathbf{x}^{s-2}\|_\infty)}.$$

例子**1** [**3**] 考虑如下非线性方程组:

$$\begin{cases} x_2x_3 + x_4(x_2+x_3) = 0, \\ x_1x_3 + x_4(x_1+x_3) = 0, \\ x_1x_2 + x_4(x_1+x_2) = 0, \\ x_1x_2 + x_1x_3 + x_2x_3 = 1. \end{cases}$$

非线性方程组的解为$\boldsymbol{\alpha} = (0.577350, 0.577350, 0.577350, -0.288675)^T$. 这里我们给出初始值为$\mathbf{x}^0 = (0.6, 0.6, 0.6, -0.2)^T$.

例子**2** [**11**] 考虑如下非线性方程组:

$$\begin{cases} e^{x_1} - x_2 - 2 = 0, \\ \cos(x_1) - x_1 + x_2 - 1 = 0. \end{cases}$$

非线性方程组的解为$\boldsymbol{\alpha} = (1.478489; 2.386312)^T$. 这里我们给出初始值为$\mathbf{x}^0 = (1.2, 2.0)^T$.

例子**3** [**3**] 考虑如下非线性方程组:

$$\begin{cases} \sin(x_1) + x_2\cos(x_1) = 0, \\ x_1 - x_2 = 0. \end{cases}$$

非线性方程组的解为$\boldsymbol{\alpha} = (0,0)^T$. 这里我们给出初始值为$\mathbf{x}^0 = (0.4, 0.4)^T$.

例子**4** [**11**] 考虑如下非线性方程组:

$$\begin{cases} 3x_1 + x_2^2 = 0, \\ x_1 - x_2(1+x_2) = 0. \end{cases}$$

非线性方程组的解为$\boldsymbol{\alpha} = (0,0)^T$. 这里我们给出初始值为$\mathbf{x}^0 = (0.5, 1)^T$.

在表格1、2和表格3、4中`IT.`, $\|\mathbf{x}^{s+1} - \mathbf{x}^s\|_\infty$, 和$\|\mathbf{f}(\mathbf{x}^s)\|_\infty$ 分别表示算法迭代步数, 算法最后两步迭代点之间的距离, 以及算法最后迭代值的残差.表格1、2与表格3、4 表示新算法1与牛顿法和牛顿-柯特斯方法作比较.例子3满足温和条件$\partial^2 f_i(\boldsymbol{\alpha})/(\partial x_j \partial x_k) = 0, i,j,k \in \{1,2\}$,定理2保证了算法的五阶收敛性.从表格中我们可以看出新算法1仍然能保持很好的收敛性质,并且相对牛顿-柯特斯方法系数要求更自由.

表 **1** 例子**1**的数值结果($\epsilon = 10^{-11}$)

	Ex.			
Alg.	IT.	$\|\mathbf{x}^{s+1} - \mathbf{x}^s\|_\infty$	$\|\mathbf{f}(\mathbf{x}^s)\|_\infty$	p
Newton	4	7.4×10^{-13}	0	2.1
NC1	3	1.6×10^{-14}	2.2×10^{-16}	3.3
NC2	3	1.6×10^{-14}	2.2×10^{-16}	3.3
算法1 ($M = 2$)	3	1.6×10^{-14}	2.2×10^{-16}	3.3
算法1 ($M = 3$)	3	1.6×10^{-14}	2.2×10^{-16}	3.3
算法1 ($M = 4$)	3	1.6×10^{-14}	2.2×10^{-16}	3.3

表 **2** 例子**2**的数值结果($\epsilon = 10^{-11}$)

	Ex.			
Alg.	IT.	$\|\mathbf{x}^{s+1} - \mathbf{x}^s\|_\infty$	$\|\mathbf{f}(\mathbf{x}^s)\|_\infty$	p
Newton	6	4.4×10^{-16}	0	1.6
NC1	4	4.4×10^{-15}	0	3.0
NC2	4	4.4×10^{-15}	0	3.0
算法1 ($M = 2$)	4	4.4×10^{-15}	0	3.0
算法1 ($M = 3$)	4	4.4×10^{-15}	0	3.0
算法1 ($M = 4$)	4	4.4×10^{-15}	0	3.0

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表 3 例子3的数值结果($\epsilon = 10^{-11}$)

	Ex.			
Alg.	IT.	$\|\mathbf{x}^{s+1} - \mathbf{x}^s\|_\infty$	$\|\mathbf{f}(\mathbf{x}^s)\|_\infty$	p
Newton	4	3.5×10^{-13}	0	3.0
NC1	3	2.2×10^{-14}	0	5.0
NC2	3	1.4×10^{-14}	0	5.0
算法1 ($M = 2$)	3	1.7×10^{-14}	0	5.0
算法1 ($M = 3$)	3	1.7×10^{-14}	0	5.0
算法1 ($M = 4$)	3	1.7×10^{-14}	0	5.0

表 4 例子4的数值结果($\epsilon = 10^{-11}$)

	Ex.			
Alg.	IT.	$\|\mathbf{x}^{s+1} - \mathbf{x}^s\|_\infty$	$\|\mathbf{f}(\mathbf{x}^s)\|_\infty$	p
Newton	7	2.1×10^{-16}	8.0×10^{-32}	2.0
NC1	5	1.5×10^{-20}	0	3.0
NC2	5	1.5×10^{-20}	0	3.0
算法1 ($M = 2$)	5	1.5×10^{-20}	0	3.0
算法1 ($M = 3$)	5	1.5×10^{-20}	0	3.0
算法1 ($M = 4$)	5	1.5×10^{-20}	0	3.0

for Solving Nonlinear Equations

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Abstract:

In this paper, we present a new variant of Newton's method for solving nonlinear equations based on Uniform coefficient Quadrature formulas. The cubic convergence of the proposed algorithm is established. Moreover, the fifth-order convergence is proved under some mild conditions. Some numerical experiments are reported to illustrate the effectiveness and flexibility of our algorithm.

Key words: Uniform coefficient quadrature, Newton-type method ,Nonlinear equations, Cubic convergence